Moments of random eigenfunctions for point scatterers on rectangular flat tori

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Berry's conjecture

M closed Riemannian manifold of negative curvature.
$(\lambda_k)_{k\geqslant 0}$ eigenvalues of Δ and $(\phi_k)_{k\geqslant 0}$ associated normalized eigenfunctions.

Notation

Let $x \in M$ and $i : \mathbb{R}^n \to T_x M$ linear isometry, we define $\phi_k^{x,i} : \mathbb{R}^n \to \mathbb{R}$ by:

$$\phi_k^{x,i} : v \longmapsto \phi_k \left(\exp_x \left(\frac{i(v)}{\sqrt{\lambda_k}} \right) \right).$$

Definition (Berry field)

$f_B : \mathbb{R}^n \to \mathbb{R}$ centered stationary Gaussian field with correlation function:

$$v \longmapsto \int_{\mathbb{S}^{n-1}} e^{i\langle v, \theta \rangle} \, \mathrm{d}\theta.$$

Berry's conjecture

Berry's conjecture (Ingremeau; Abert–Bergeron–Le Masson)

Let X be a uniform random point in M, and let I be a uniform random variable in $\mathrm{Isom}(\mathbb{R}^n, T_X M) \simeq O_n(\mathbb{R})$. Then

$$\phi_k^{X,I} \xrightarrow[k\to+\infty]{d} f_B.$$

Corollary

$$\phi_k(X) = \phi_k^{X,I}(0) \xrightarrow[k\to+\infty]{d} f_B(0) \sim \mathcal{N}(0,1)$$

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Moments conjecture

For all $p \in \mathbb{N}^*$, $\mathbb{E}[\phi_k(X)^p] \xrightarrow[k\to+\infty]{} \mu_p$.

Point scatterers on flat tori

Spectrum of the Laplacian on flat tori

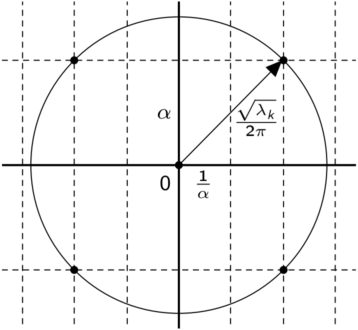
Let $\alpha > 0$, we denote $\mathbb{T}_\alpha = \mathbb{R}^2 / \left(\alpha\mathbb{Z} \oplus \frac{1}{\alpha}\mathbb{Z}\right)$.

$$\mathrm{Sp}(\Delta) = \left\{ 4\pi^2 \left(\frac{a^2}{\alpha^2} + \alpha^2 b^2 \right) \,\middle|\, a, b \in \mathbb{N} \right\} = \{\lambda_k \mid k \geqslant 0\}.$$

$$\Lambda_k = \left\{ \xi \in \tfrac{1}{\alpha}\mathbb{Z} \oplus \alpha\mathbb{Z} \,\middle|\, 4\pi^2 \|\xi\|^2 = \lambda_k \right\}.$$

An orthonormal basis of $\ker(\Delta - \lambda_k \, \mathrm{Id})$ is $\left\{ e^{2i\pi\langle \xi \,, \cdot \rangle} \,\middle|\, \xi \in \Lambda_k \right\}$.

$r_k = \#\Lambda_k$.



Two examples

Irrational tori ($\alpha^4 \notin \mathbb{Q}$)

We have $r_k \in \{1, 2, 4\}$, and $r_k = 4$ along a density 1 subsequence.

Square torus ($\alpha = 1$)

$r_k = 8$ for inifinitely many k. On average, r_k is of order $\sqrt{\ln(\lambda_k)}$:

$$\frac{1}{k+1} \sum_{i=0}^{k} r_i \sim C\sqrt{\ln(\lambda_k)}.$$

Point scatterers on $\mathbb{T}_\alpha$

A point scatterer is an unbounded self-adjoint operator on $L^2(\mathbb{T}_\alpha)$ of the form "$\Delta + C\delta_0$".

Theorem (von Neumann)

Let $D_0 = \mathcal{C}_c^\infty(\mathbb{T}_\alpha \setminus \{0\})$, there exists a one-parameter family $(\Delta_\varphi)_{\varphi \in (-\pi,\pi]}$ of self-adjoint extensions of $\Delta_{|D_0}$.

For $\varphi = \pi$, we recover the usual Laplacian.

A *point scatterer* on $\mathbb{T}_\alpha$ is an element of the family $(\Delta_\varphi)_{\varphi \in (-\pi,\pi)}$.

Spectrum of Δ_φ ($\varphi \neq \pi$)

For all $k \geqslant 1$, λ_k is an eigenvalue of Δ_φ of multiplicity $r_k - 1$ and

$$\ker\left(\Delta_\varphi - \lambda_k \,\mathrm{Id}\right) = \left\{\phi \in \ker(\Delta - \lambda_k \,\mathrm{Id}) \mid \phi(0) = 0\right\}.$$

The new eigenvalues of Δ_φ are simple and can be ordered in an increasing sequence $(\tau_n)_{n \geqslant 0}$ such that:

$$\tau_0 < \lambda_0 = 0 < \tau_1 < \lambda_1 < \cdots < \tau_n < \lambda_n < \ldots.$$

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For all $\tau \in \mathbb{R} \setminus \mathrm{Sp}(\Delta)$, let $f_\tau = \sum_{k \geqslant 1} \sum_{\xi \in \Lambda_k} \frac{e^{2i\pi\langle \xi , \cdot \rangle}}{\lambda_k - \tau}$ and $G_\tau = -\frac{1}{\tau} + f_\tau$.

G_{τ_n} is an eigenfunction of Δ_φ associated with τ_n.

Moments conjecture for point scatterers

Definition

For $p \in \mathbb{N}^*$ and $\tau \in \mathbb{R} \setminus \mathrm{Sp}(\Delta)$, let $M_\tau^p = \int_{\mathbb{T}_\alpha} f_\tau(x)^p \, \mathrm{d}x$.

We have $M_\tau^1 = 0$ and $M_\tau^2 = \sum_{k \geqslant 1} \frac{r_k}{(\lambda_k - \tau)^2}$.

Question

Do we have: for all $p \geqslant 1$,

$$\frac{M_\tau^p}{(M_\tau^2)^{\frac{p}{2}}} \xrightarrow[\tau \to +\infty]{} \mu_p ?$$

Random model for the moments of the new eigenfunctions

Reminder

Definition

Let ν be a measure on $[0, +\infty)$, a Poisson Point Process of intensity ν is a random subset $P \subset [0, +\infty)$ such that:

- For any interval I, $\#(I \cap P)$ is a Poisson variable of parameter $\nu(I)$.
- If $I_1, \ldots, I_n$ are disjoint intervals, $(\#(I_k \cap P))_{1 \leqslant k \leqslant n}$ are independent.

Theorem (Weyl Law)

On $\mathbb{T}_\alpha$, we have:

$$\sum_{\lambda_k \leqslant \lambda} r_k = \frac{\lambda}{4\pi} + O(\sqrt{\lambda}).$$

The Berry–Tabor conjecture

Conjecture

The sequence $(\lambda_k)_{k\geqslant 1}$ "behaves like" a Poisson Point Process on $[0,+\infty)$.

- Numerics on irrational tori: $\frac{1}{N}\sum_{k=1}^{N}\delta_{\lambda_k-\lambda_{k-1}} \xrightarrow[N\to+\infty]{d} \mathcal{E}\left(\frac{1}{16\pi}\right)$.
- Sarnak: $\frac{1}{N}\sum_{1\leqslant k,l\leqslant N}\delta_{\lambda_k-\lambda_l}$ has a Poissonian limit for a.e. flat torus.
- Freiberg–Kurlberg–Rosenzweig: conditional result on the square torus.

A simple plan

- Replace $(\lambda_k)_{k\geqslant 1}$ by a Poisson Point Process on $[0,+\infty)$ whose intensity is consistent with the Weyl Law.

- For each $k \geqslant 1$, define $\frac{2\pi}{\sqrt{\lambda_k}}\Lambda_k$ as a random subset of $\mathbb{S}^1$, invariant by symmetry with respect to the coordinate axes.

Step 1: expression of the deterministic moments

Notation

- $\ell_0 = \left\{(a_k)_{k\geqslant 1} \in \mathbb{N}^{\mathbb{N}^*} \;\middle|\; \sum_{k\geqslant 1} a_k < +\infty\right\}$.
- For $a = (a_k) \in \ell_0$, we denote $|a| = \sum_{k\geqslant 1} a_k$ and $a! = \prod_{k\geqslant 1} a_k!$.

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Lemma

Let $p \geqslant 1$ and $\tau \in \mathbb{R} \setminus \mathrm{Sp}(\Delta)$, we have:

$$M_\tau^p = p! \sum_{a \in \ell_0 ; |a| = p} \frac{N_a}{a!} \prod_{k \geqslant 1} \left(\frac{1}{\lambda_k - \tau} \right)^{a_k},$$

where

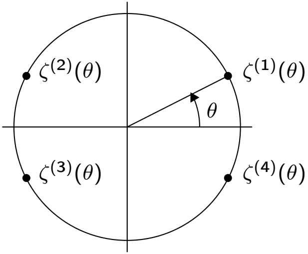
$$N_a = \# \left\{ (\xi_{k,l})_{\substack{k \geqslant 1 \\ 1 \leqslant l \leqslant a_k}} \middle| \forall (k,l), \xi_{k,l} \in \Lambda_k \text{ and } \sum_{k \geqslant 1} \sum_{l=1}^{a_k} \xi_{k,l} = 0 \right\}.$$

Step 2: randomization of the wave vectors

Assume we are given $(\lambda_k)_{k\geqslant 1}$ an increasing sequence of positive numbers and $(m_k)_{k\geqslant 1}$ a sequence of positive integers.

Notation

η distribution of $(\theta_{k,j})_{k,j\geqslant 1}$, where $\theta_{k,j}$ are independent uniform in $[0, \frac{\pi}{2}]$.



Let $(\theta_{k,j})_{k,j\geqslant 1}$ be random variables in $[0, \frac{\pi}{2}]$ whose distribution admits a density with respect to η. For all $k \geqslant 1$, we set:

$$\Lambda_k = \frac{\sqrt{\lambda_k}}{2\pi} \cdot \left\{ \zeta^{(i)}(\theta_{k,j}) \,\middle|\, 1 \leqslant i \leqslant 4 \text{ and } 1 \leqslant j \leqslant m_k \right\}.$$

Almost sure results

Almost surely, for all $k \geqslant 1$, $r_k = \#\Lambda_k = 4m_k$.

Almost sure expression of $(N_a)_{a\in\ell_0}$:

- depends only on $(m_k)_{k\geqslant 1}$;
- if $a = (a_k)_{k\geqslant 1}$ and $\exists k_0$ s.t. a_{k_0} is odd, then $N_a = 0$.

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Lemma

Almost surely, for all $p \geqslant 1$ and all $\tau \notin \{\lambda_k \mid k \geqslant 1\}$, we have $M_\tau^{2p-1} = 0$ and:

$$M_\tau^{2p} = (2p)! \sum_{\substack{\alpha\in\ell_0 \\ \sum q\alpha_q = p}} \frac{(-1)^{p-|\alpha|}}{\alpha!} \prod_{q=1}^{p} (A_q S_\tau^q)^{\alpha_q},$$

where for all $q \geqslant 1$, $A_q > 0$ and $S_\tau^q = \displaystyle\sum_{k\geqslant 1} \frac{m_k}{(\lambda_k - \tau)^{2q}}$.

Step 3: randomization of the spectrum

Definition (Multiplicity function)

A multiplicity function is a $\mathcal{C}^1$ function $m : [0, +\infty) \to [1, +\infty)$ s.t. $m'(t) = O(t^{-\beta})$ as $t \to +\infty$, for some $\beta > 0$.

- Choose a multiplicity function m.
- Model the spectrum by the values $(\lambda_k)_{k \geqslant 1}$ of a Poisson Point Process on $[0, +\infty)$ of intensity $\frac{1}{16\pi m(t)}\, \mathrm{d}t$.
- Set $m_k = m(\lambda_k)$, for all $k \geqslant 1$.

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Examples

- $m : t \mapsto 1$ (irrational tori).
- $m : t \mapsto 1 + \sqrt{\ln(1+t)}$ (square torus).
- $m : t \mapsto 1 + t^\alpha$ with $\alpha < 1$.

Probabilistic Weyl Law

For $\lambda \geqslant 0$, let $N(\lambda) = 4 \sum_{\lambda_k \leqslant \lambda} m(\lambda_k)$. Then we have: $\mathbb{E}[N(\lambda)] = \frac{\lambda}{4\pi}$.

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Proposition

- *Almost surely:* $\frac{1}{\lambda} N(\lambda) \xrightarrow[\lambda \to +\infty]{} \frac{1}{4\pi}$.
- *We have* $\mathrm{Var}(N(\lambda)) = \frac{1}{\pi} \int_0^\lambda m(t) \, \mathrm{d}t$ *and:*

$$\frac{N(\lambda) - \mathbb{E}[N(\lambda)]}{\sqrt{\mathrm{Var}(N(\lambda))}} \xrightarrow[\lambda \to +\infty]{d} \mathcal{N}(0, 1).$$

Randomized even moments

The random variables

$$S_\tau^q = \sum_{k \geqslant 1} \frac{m(\lambda_k)}{(\lambda_k - \tau)^{2q}}$$

and

$$M_\tau^{2p} = (2p)! \sum_{\substack{\alpha \in \ell_0 \\ \sum q\alpha_q = p}} \frac{(-1)^{p-|\alpha|}}{\alpha!} \prod_{q=1}^{p} (A_q S_\tau^q)^{\alpha_q}$$

are almost surely well-defined.

Theorem (L.–Ueberschär)

Let $(\tau_n)_{n\geqslant 0}$ be a sequence of real numbers such that $\tau_n \xrightarrow[n\to+\infty]{} +\infty$ and $m(\tau_n) \xrightarrow[n\to+\infty]{} \ell \in [1,+\infty]$. Then, for all $p \geqslant 1$,

$$\left(\frac{M_{\tau_n}^{2q}}{\left(M_{\tau_n}^{2}\right)^{q}}\right)_{1\leqslant q\leqslant p} \xrightarrow[n\to+\infty]{d} (\mu_2, \mu_4 R_2(\ell), \ldots, \mu_{2p} R_p(\ell))\,,$$

where $\mu_{2q} = \frac{(2q)!}{2^q q!}$ and $(R_2(\ell), \ldots, R_p(\ell))$ is a random vector in $\mathbb{R}^{p-1}$.

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$$\left(\frac{M_{\tau_n}^{2q}}{\left(M_{\tau_n}^{2}\right)^q}\right)_{1\leqslant q\leqslant p} \xrightarrow[n\to+\infty]{d} \left(\mu_2, \mu_4 R_2(\ell), \dots, \mu_{2p} R_p(\ell)\right),$$

where $\mu_{2q} = \frac{(2q)!}{2^q q!}$ and $(R_2(\ell), \dots, R_p(\ell))$ is a random vector in $\mathbb{R}^{p-1}$.

- If $\ell = +\infty$, $(R_2(\ell), \dots, R_p(\ell)) = (1, \dots, 1)$ almost surely.
- If $\ell < +\infty$, then $(R_2(\ell), \dots, R_p(\ell))$ admits a smooth density $\mathcal{D}_\ell$ with respect to Lebesgue.
- If $\ell \neq \ell'$, then $\mathcal{D}_\ell \neq \mathcal{D}_{\ell'}$.

Main tool

Theorem (Campbell)

Let $(\lambda_k)_{k\geqslant 1}$ *be a Poisson Point Process on* $[0,+\infty)$ *of intensity* ν*. For any* $q\in\{1,\dots,p\}$*, let* $g_q:[0,+\infty)\to[0,+\infty)$ *be a measurable map such that*

$$\int_0^{+\infty}\min(g_q,1)\,\mathrm{d}\nu<+\infty,$$

and let $S^q=\sum_{k\geqslant 1}g_q(\lambda_k)$*. Then, for all* $(x_1,\dots,x_p)\in\mathbb{R}^p$*,*

$$\mathbb{E}\left[\exp\left(i\sum_{q=1}^{p}x_qS^q\right)\right]=\exp\left(-\int_0^{+\infty}\left(1-\exp\left(i\sum_{q=1}^{p}x_qg_q\right)\right)\mathrm{d}\nu\right).$$

Idea of the proof

Using Campbell's Theorem, for any choice of the multiplicity function m,

$$(m(\tau)S_\tau^1, m(\tau)^3 S_\tau^2, \dots, m(\tau)^{2p-1} S_\tau^p) \xrightarrow[\tau\to+\infty]{d} S = (S^1, \dots, S^p),$$

where the characteristic function of S is:

$$\psi : (x_1, \dots, x_p) \mapsto \exp\left(-\frac{1}{16\pi}\int_{-\infty}^{+\infty} 1 - \exp\left(i\sum_{q=1}^{p} \frac{x_q}{t^{2q}}\right) \mathrm{d}t\right).$$

Lemma

The function ψ is fast decreasing, hence S admits a smooth density.

Apply the Continuous Mapping Theorem.